

# The deformations of antibracket with even and odd deformation parameters

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## Abstract

We consider antiPoisson superalgebras realized on the smooth Grassmann-valued functions with compact supports in  $\mathbb{R}^n$  and with the grading inverse to Grassmanian parity. The deformations with even and odd deformation parameters of these superalgebras are presented for arbitrary  $n$ .

## 1 Introduction

In [10] we described the deformation of Poisson superalgebra depending on even and finite number of odd deformation parameters. The number of finite parameters in that case may be arbitrary because Poisson superalgebra realized on the smooth Grassmann-valued functions with compact support has infinite number of odd 2-cocycles in adjoint representation.

Here we consider the deformations of antiPoisson superalgebras realized on the smooth Grassmann-valued functions with compact supports in  $\mathbb{R}^n$  and show that there exists either one deformation with one even deformation parameter, or one deformation with one odd parameter.

All necessary definitions are in the next section. This text organized as follows.

Section 3 contains previously known results about second cohomology space of antibracket and two more cohomologies for  $n = 1$ . Theorem 4.3 described the general form of the deformations are formulated in Section 4 and proved in Section 7. Cohomology space  $H_{\mathbf{E}}^2$  is described in Section 6 and with details in Appendix 1.

## 2 General

The odd Poisson bracket play an important role in Lagrangian formulation of the quantum theory of the gauge fields, which is known as BV-formalism [1], [2] (see also [3]-[5]). These odd

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bracket were introduced in physical literature in [1] and were called there as "antibracket". Antibracket possesses many features analogous to ones of even Poisson bracket and even can be obtained via "canonical formalism" with odd time. However, contrary to the case of even Poisson bracket where there exists voluminous literature on different aspects of the deformation (quantization) of Poisson algebra, the problem of the deformation of antibracket is not study satisfactory yet.

In [6] the deformations antibracket realized on the space of vector fields with polynomial coefficients are found and in [9] the deformation of antibracket realized on the smooth Grassmann-valued functions with compact support is found.

The goal of present work is finding all the deformations depending on even and odd deformation parameters of antiPoisson superalgebra realized on the smooth Grassmann-valued functions with compact supports in  $\mathbb{R}^n$ .

Let  $\mathbb{K}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . We denote by  $\mathcal{D}(\mathbb{R}^n)$  the space of smooth  $\mathbb{K}$ -valued functions with compact supports on  $\mathbb{R}^n$ . This space is endowed with its standard topology. We set

$$\mathbf{D}_n = \mathcal{D}(\mathbb{R}^n) \otimes \mathbb{G}^n, \quad \mathbf{E}_n = C^\infty(\mathbb{R}^n) \otimes \mathbb{G}, \quad \mathbf{D}'_n = \mathcal{D}'(\mathbb{R}^n) \otimes \mathbb{G}^n,$$

where  $\mathbb{G}^n$  is the Grassmann algebra with  $n$  generators and  $\mathcal{D}'(\mathbb{R}^n)$  is the space of continuous linear functionals on  $\mathcal{D}(\mathbb{R}^n)$ . The generators of the Grassmann algebra (resp., the coordinates of the space  $\mathbb{R}^n$ ) are denoted by  $\xi^\alpha$ ,  $\alpha = 1, \dots, n$  (resp.,  $x^i$ ,  $i = 1, \dots, n$ ). We shall also use collective variables  $z^A$  which are equal to  $x^A$  for  $A = 1, \dots, n$  and are equal to  $\xi^{A-n}$  for  $A = n+1, \dots, 2n$ .

The spaces  $\mathbf{D}_n$ ,  $\mathbf{E}_n$ , and  $\mathbf{D}'_n$  possess a natural grading which is determined by that of the Grassmann algebra. The Grassmann parity ( $\varepsilon$ -parity) of an element  $f$  of these spaces is denoted by  $\varepsilon(f)$ .

The spaces  $\mathbf{D}_n$ ,  $\mathbf{E}_n$ , and  $\mathbf{D}'_n$  possess also another  $\mathbb{Z}_2$ -grading  $\epsilon$  ( $\epsilon$ -parity), which is inverse to  $\varepsilon$ -parity:  $\epsilon = \varepsilon + 1$ .

We set  $\varepsilon_A = 0$ ,  $\epsilon_A = 1$  for  $A = 1, \dots, n$  and  $\varepsilon_A = 1$ ,  $\epsilon_A = 0$  for  $A = n+1, \dots, 2n$ .

It is well known, that the bracket

$$[f, g](z) = \sum_{i=1}^n \left( f(z) \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\partial}{\partial \xi^i} g(z) - f(z) \frac{\overleftarrow{\partial}}{\partial \xi^i} \frac{\partial}{\partial x^i} g(z) \right), \quad (2.1)$$

which we will call "antibracket", defines the structure of Lie superalgebra on the superspaces  $\mathbf{D}_n$  and  $\mathbf{E}_n$  with the  $\epsilon$ -parity.

Indeed,  $[f, g] = -(-1)^{\epsilon(f)\epsilon(g)}[g, f]$ ,  $\epsilon([f, g]) = \epsilon(f) + \epsilon(g)$ , and Jacobi identity is satisfied:

$$(-1)^{\epsilon(f)\epsilon(h)}[f, [g, h]] + (-1)^{\epsilon(g)\epsilon(f)}[g, [h, f]] + (-1)^{\epsilon(h)\epsilon(g)}[h, [f, g]] = 0, \quad f, g, h \in \mathbf{E}_n. \quad (2.2)$$

Evidently, the metric  $\omega$  defining antibracket

$$[f, g](z) = f(z) \frac{\overleftarrow{\partial}}{\partial z^A} \omega^{AB} \frac{\partial}{\partial z^B} g(z),$$

is constant, nondegenerate, and satisfy the condition

$$\omega^{BA} = -(-1)^{\epsilon_A \epsilon_B} \omega^{AB}, \quad \epsilon(\omega^{AB}) = \epsilon_A + \epsilon_B,$$

Here these Lie superalgebras are called antiPoisson superalgebras.<sup>1</sup>

The integral on  $\mathbf{D}_n$  is defined by the relation  $\int dz f(z) = \int_{\mathbb{R}^n} dx \int d\xi f(z)$ , where the integral on the Grassmann algebra is normed by the condition  $\int d\xi \xi^1 \dots \xi^n = 1$ . We identify  $\mathbb{G}^n$  with its dual space  $\mathbb{G}'^n$  setting  $f(g) = \int d\xi f(\xi)g(\xi)$ ,  $f, g \in \mathbb{G}^n$ . Correspondingly, the space  $\mathbf{D}'_n$  of continuous linear functionals on  $\mathbf{D}_n$  is identified with the space  $\mathcal{D}'(\mathbb{R}^n) \otimes \mathbb{G}^n$ . The value  $m(f)$  of a functional  $m \in \mathbf{D}'_n$  on a test function  $f \in \mathbf{D}_n$  will be often written in the integral form:  $m(f) = \int dz m(z)f(z)$ .

### 3 Cohomology of antibrackets (Results)

Let  $\mathbf{D}_n$  acts in a  $\mathbb{Z}_2$ -graded space  $V$  (the action of  $f \in \mathbf{D}_n$  on  $v \in V$  will be denoted by  $f \cdot v$ ). The space  $C_p(\mathbf{D}_n, V)$  of  $p$ -cochains consists of all multilinear superantisymmetric mappings from  $\mathbf{D}_n^p$  to  $V$ . Superantisymmetry means, as usual, that  $M_p(\dots, f_i, f_{i+1}, \dots) = -(-1)^{\epsilon(f_i)\epsilon(f_{i+1})} M_p(\dots, f_{i+1}, f_i, \dots)$ . The space  $C_p(\mathbf{D}_n, V)$  possesses a natural  $\mathbb{Z}_2$ -grading: by definition,  $M_p \in C_p(\mathbf{D}_n, V)$  has the definite  $\epsilon$ -parity  $\epsilon_{M_p}$  if

$$\epsilon(M_p(f_1, \dots, f_p)) = \epsilon_{M_p} + \epsilon(f_1) + \dots + \epsilon(f_p)$$

for any  $f_j \in \mathbf{D}_n$  with  $\epsilon$ -parities  $\epsilon(f_j)$ . We will often use the Grassmann  $\varepsilon$ -parity<sup>2</sup> of cochains:  $\varepsilon_{M_p} = \epsilon_{M_p} + p + 1$ . The differential  $d_p^V$  is defined to be the linear operator from  $C_p(\mathbf{D}_n, V)$  to  $C_{p+1}(\mathbf{D}_n, V)$  such that

$$\begin{aligned} d_p^V M_p(f_1, \dots, f_{p+1}) = & - \sum_{j=1}^{p+1} (-1)^{j+\epsilon(f_j)|\epsilon(f)|_{1,j-1}+\epsilon(f_j)\epsilon_{M_p}} f_j \cdot M_p(f_1, \dots, \check{f}_j, \dots, f_{p+1}) - \\ & - \sum_{i < j} (-1)^{j+\epsilon(f_j)|\epsilon(f)|_{i+1,j-1}} M_p(f_1, \dots, f_{i-1}, [f_i, f_j], f_{i+1}, \dots, \check{f}_j, \dots, f_{p+1}), \end{aligned} \quad (3.1)$$

for any  $M_p \in C_p(\mathbf{D}_n, V)$  and  $f_1, \dots, f_{p+1} \in \mathbf{D}_n$  having definite  $\epsilon$ -parities. Here the sign  $\check{\phantom{x}}$  means that the argument is omitted and the notation

$$|\epsilon(f)|_{i,j} = \sum_{l=i}^j \epsilon(f_l)$$

has been used. The differential  $d^V$  is nilpotent (see [7]), i.e.,  $d_{p+1}^V d_p^V = 0$  for any  $p = 0, 1, \dots$ . The  $p$ -th cohomology space of the differential  $d_p^V$  will be denoted by  $H_p^V$ . The second cohomology space  $H_{\text{ad}}^2$  in the adjoint representation is closely related to the problem of finding formal deformations of the Lie bracket  $[\cdot, \cdot]$  of the form  $[f, g]_* = [f, g] + \hbar[f, g]_1 + \dots$  up to similarity transformations  $[f, g]_T = T^{-1}[Tf, Tg]$  where continuous linear operator  $T$  from  $V[[\hbar]]$  to  $V[[\hbar]]$  has the form  $T = \text{id} + \hbar T_1$ .

<sup>1</sup> We will consider usual multiplication of the elements of considered antiPoisson superalgebras with commutation relations  $fg = (-1)^{\varepsilon(f)\varepsilon(g)}gf$  as well, and the variables  $x^i$  will be called even variables and the variables  $\xi^i$  will be called odd variables.

<sup>2</sup> If  $V$  is the space of Grassmann-valued functions on  $\mathbb{R}^n$  then  $\varepsilon$  defined in such a way coincides with usual Grassmann parity.

The condition that  $[\cdot, \cdot]_1$  is a 2-cocycle is equivalent to the Jacobi identity for  $[\cdot, \cdot]_*$  modulo the  $\hbar$ -order terms.

In the present paper, similarly to [8], we suppose that cochains are separately continuous multilinear mappings.

We need the cohomologies of the antiPoisson algebra  $\mathbf{D}_n$  in the following representations:

1.  $V = \mathbf{E}_n$  and  $f \cdot g = [f, g]$  for any  $f \in \mathbf{D}_n, g \in \mathbf{E}_n$ . The space  $C_p(\mathbf{D}_n, \mathbf{E}_n)$  consists of separately continuous superantisymmetric multilinear mappings from  $(\mathbf{D}_n)^p$  to  $\mathbf{E}_n$ . The cohomology spaces and the differentials will be denoted by  $H_{\mathbf{E}}^p$  and  $d_p^{\text{ad}}$  respectively.
2. The adjoint representation:  $V = \mathbf{D}_n$  and  $f \cdot g = [f, g]$  for any  $f, g \in \mathbf{D}_n$ . The space  $C_p(\mathbf{D}_n, \mathbf{D}_n)$  consists of separately continuous superantisymmetric multilinear mappings from  $(\mathbf{D}_n)^p$  to  $\mathbf{D}_n$ . The cohomology spaces and the differentials will be denoted by  $H_{\text{ad}}^p$  and  $d_p^{\text{ad}}$  respectively.

We shall call p-cocycles  $M_p^1, \dots, M_p^k$  independent cohomologies if they give rise to linearly independent elements in  $H^p$ . For a multilinear form  $M_p$  taking values in  $\mathbf{D}_n, \mathbf{E}_n$ , or  $\mathbf{D}'_n$ , we write  $M_p(z|f_1, \dots, f_p)$  instead of more cumbersome  $M_p(f_1, \dots, f_p)(z)$ .

The following theorems proved in [9] describe these cohomology of antibracket

**Theorem 3.1.** *Let the bilinear mappings  $m_{2|1}, m_{2|2}, m_{2|5}, m_{2|6}$  from  $(\mathbf{D}_1)^2$  to  $\mathbf{E}_1$  and bilinear mappings  $m_{2|3}, m_{2|4}$  from  $(\mathbf{D}_n)^2$  to  $\mathbf{E}_n$  be defined by the relations*

$$m_{2|1}(z|f, g) = \int du \partial_\eta g(u) \partial_y^3 f(u), \quad \epsilon_{m_{2|1}} = 1, \quad (3.2)$$

$$m_{2|2}(z|f, g) = \int du \theta(x - y) [\partial_\eta g(u) \partial_y^3 f(u) - \partial_\eta f(u) \partial_y^3 g(u)] + \\ + x [\{\partial_\xi \partial_x^2 f(z)\} \partial_\xi \partial_x g(z) - \{\partial_\xi \partial_x f(z)\} \partial_\xi \partial_x^2 g(z)], \quad \epsilon_{m_{2|2}} = 1, \quad (3.3)$$

$$m_{2|3}(z|f, g) = (-1)^{\epsilon(f)} \{(1 - N_\xi) f(z)\} (1 - N_\xi) g(z), \quad \epsilon_{m_{2|3}} = 1, \quad (3.4)$$

$$m_{2|4}(z|f, g) = (-1)^{\epsilon(f)} \{\Delta f(z)\} \mathcal{E}_z g(z) + \{\mathcal{E}_z f(z)\} \Delta g(z) \quad \epsilon_{m_{2|4}} = 0. \quad (3.5)$$

$$m_{2|5}(z|f, g) = \int du (-1)^{\epsilon(f)} \partial_y f(u) \partial_y g(u), \quad \epsilon_{m_{2|5}} = 0, \quad (3.6)$$

$$m_{2|6}(z|f, g) = \int du \theta(x - y) (-1)^{\epsilon(f)} \partial_y f(u) \partial_y g(u), \quad \epsilon_{m_{2|6}} = 0 \quad (3.7)$$

where  $z = (x, \xi)$ ,  $u = (y, \eta)$ ,  $N_\xi = \xi \partial_\xi$ , and

$$\Delta = \partial_x \partial_\xi. \quad (3.8)$$

Then

1.  $H_{\text{ad}}^2 \simeq \mathbb{K}^2$  and the cochains  $m_{2|3}(z|f, g)$  and  $m_{2|4}(z|f, g)$  are independent nontrivial cocycles.

2. Let  $n = 1$ .

Then  $H_{\mathbf{E}}^2 \simeq \mathbb{K}^6$  and the cochains  $m_{2|1}(z|f, g)$ ,  $m_{2|2}(z|f, g)$ ,  $m_{2|3}(z|f, g)$ ,  $m_{2|4}(z|f, g)$ ,  $m_{2|5}(z|f, g)$ , and  $m_{2|6}(z|f, g)$  are independent nontrivial cocycles.

3. Let  $n \geq 2$ . Then  $H_{\mathbf{D}}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^2$  and the cochains  $m_{2|3}(z|f, g)$  and  $m_{2|4}(z|f, g)$  are independent nontrivial cocycles.

Nonlocal cocycles  $m_{2|5}$  and  $m_{2|6}$  are lost in [9] for  $n = 1$ . This is the reason to reproduce below (in Section 6 and in Appendix 1) the proof of item 2 and item 1 for  $n = 1$  of Theorem 3.1.

## 4 Deformations of antibrackets (Results)

Consider general form of deformation,  $[f, g]_*(z)$ , of the antibracket  $[f, g](z)$ .

Because antibracket on  $\mathbf{D}_n$  has only two independent adjoint second cohomology, one even ( $m_{2|4}$ ) and one odd ( $m_{2|3}$ ), we consider the deformations depending on one even and one odd (in Grassmannian sense) parameters,  $\hbar$  and  $\theta$  correspondingly.

We will suppose that:

1.

$$[f, g]_*(z) \equiv C(z|f, g; \hbar, \theta) = A_1(z|f, g; \hbar) + \theta A_0(z|f, g; \hbar), \quad \epsilon(A_i) = i + 1 \quad (4.1)$$

$$A_1(z|f, g; 0) = [f, g](z), \quad (4.2)$$

2.

$$A_i(z|f, g; \hbar) = \sum_k \hbar^k A_{i|k}(z|f, g)$$

3.  $A_{i|k}(z|f, g) \in \mathbf{D}_n$ , for all  $f, g \in \mathbf{D}_n$ ;

4.  $[f, g]_*(z)$  satisfies the Jacobi identity

$$(-1)^{\epsilon(f)\epsilon(h)}[[f, g]_*, h]_* + \text{cycle}(f, g, h) = 0, \quad \forall f, g, h \in \mathbf{D}_n, \quad (4.3)$$

or

$$(-1)^{\epsilon(f)\epsilon(h)}C(z|C(|f, g), h) + \text{cycle}(f, g, h) = 0. \quad (4.4)$$

Note that if a form  $C(z|f, g)$  satisfies the Jacobi identity then the form  $C_T(z|f, g)$ ,

$$C_T(z|f, g) = T^{-1}C(z|Tf, Tg),$$

satisfies the Jacobi identity too. Here  $T: f(z) \rightarrow T(z|f)$  is invertible continuous map  $\mathbf{D}_n \rightarrow \mathbf{D}_n$ .

Formal deformations  $C^1$  and  $C^2$  are called similar if there is a continuous  $\mathbb{K}[[\hbar, \theta]]$ -linear parity conserving similarity operator  $T: \mathbf{D}_n[[\hbar, \theta]] \rightarrow \mathbf{D}_n[[\hbar, \theta]]$  such that  $TC^1(f, g) = C^2(Tf, Tg)$ ,  $f, g \in \mathbf{D}_n[[\hbar, \theta]]$  and  $T = id + T_1$ , where  $T_1 = 0$  if  $\hbar = 0$  and  $\theta = 0$ .

Theorem 3.1 allows us to prove the following theorem, stating the general form of the deformation of antiPoisson superalgebra with even deformation parameter:

**Theorem 4.1.** [9] *The deformation of antiPoisson superalgebra with even parameter  $\hbar$  has the form*

$$[f(z), g(z)]_* = [f(z), g(z)] + (-1)^{\varepsilon(f)} \left\{ \frac{\hbar c}{1 + \hbar c N_z / 2} \Delta f(z) \right\} \mathcal{E}_z g(z) + \{ \mathcal{E}_z f(z) \} \frac{\hbar c}{1 + \hbar c N_z / 2} \Delta g(z) \quad (4.5)$$

up to similarity transformation, where  $N_z = z^A \frac{\partial}{\partial z^A}$ , and  $c$  is an arbitrary formal series in  $\hbar$  with coefficients in  $\mathbb{K}$ .

The identity  $\theta^2 = 0$  and Theorem 3.1 lead to evident result:

**Theorem 4.2.** *The deformation of antiPoisson superalgebra with odd parameter  $\theta$  has the form*

$$[f(z), g(z)]_* = [f(z), g(z)] + \theta((-1)^{\varepsilon(f)} \{ \Delta f(z) \} \mathcal{E}_z g(z) + \{ \mathcal{E}_z f(z) \} \Delta g(z)) \quad (4.6)$$

Main result of present work is the following theorem which is proved below

**Theorem 4.3.** *The deformation of antiPoisson superalgebra with one even and odd parameters has either the form (4.5) or the form (4.6).*

## 5 Preliminary and Notation

We define  $\delta$ -function by the formula

$$\int dz' \delta(z' - z) f(z') = \int f(z') \delta(z - z') dz' = f(z).$$

Evidently,

$$[f, g](z) = (-1)^{\varepsilon_A \varepsilon(f)} \frac{\partial}{\partial z^A} (f(z) \omega^{AB} \frac{\partial}{\partial z^B} g(z)) - 2f \Delta g(z),$$

$$(-1)^{\varepsilon(g)} \int dz f[g, h] = \int dz [f, g] h + 2 \int dz f \Delta g h,$$

where  $\Delta$  is defined by (3.8).

The following notation is used below:

$$\begin{aligned} T_{\dots(A)_k \dots} &\equiv T_{\dots A_1 \dots A_k \dots}, & T_{\dots A_i A_{i+1} \dots} &= (-1)^{\varepsilon_{A_i} \varepsilon_{A_{i+1}}} T_{\dots A_{i+1} A_i \dots}, & i &= 1, \dots, k-1 \\ T_{\dots(A)_k \dots} Q_{\dots}^{(A)_k} &\dots \equiv T_{\dots A_1 \dots A_k \dots} Q_{\dots}^{A_1 \dots A_k \dots}, \\ (\partial_A)^Q &\equiv \partial_{A_1} \partial_{A_2} \dots \partial_{A_Q}, & (p_A)^Q &\equiv p_{A_1} p_{A_2} \dots p_{A_Q}, \end{aligned}$$

and so on.

We denote by  $M_p(\dots)$  the separately continuous superantisymmetrical  $p$ -linear forms on  $(\mathbf{D}_n)^p$ . Thus, the arguments of these functionals are the functions  $f(z)$  of the form

$$f(z) = \sum_{k=0}^n f_{(\alpha)_k}(x) (\xi^\alpha)^k \in \mathbf{D}_n, \quad f_{(\alpha)_k}(x) \in \mathcal{D}(\mathbb{R}^n). \quad (5.1)$$

For any  $f(z) \in \mathbf{D}_n$  we can define the support

$$\text{supp}(f) \stackrel{\text{def}}{=} \bigcup_{(\alpha)_k} \text{supp}(f_{(\alpha)_k}(x)).$$

For each set  $V \subset \mathbb{R}^n$  we use the notation  $z \cap V = \emptyset$  if  $z = (x, \xi)$  and there exist some domain  $U \subset \mathbb{R}^n$  such that  $x \in U$  and  $U \cap V = \emptyset$ .

It can be easily proved that such multilinear forms can be written in the integral form (see [8]):

$$M_p(f_1, \dots, f_p) = \int dz_p \cdots dz_1 m_p(z_1, \dots, z_p) f_1(z_1) \cdots f_p(z_p), \quad p = 1, 2, \dots \quad (5.2)$$

and

$$M_p(z|f_1, \dots, f_p) = \int dz_p \cdots dz_1 m_p(z|z_1, \dots, z_p) f_1(z_1) \cdots f_p(z_p), \quad p = 1, 2, \dots \quad (5.3)$$

Let by definition

$$\epsilon(M_p(f_1, \dots, f_p)) = \epsilon_{m_p} + pn + \epsilon(f_1) + \dots + \epsilon(f_p).$$

It follows from the properties of the forms  $M_p$  that the corresponding kernels  $m_p$  have the following properties:

$$\begin{aligned} \epsilon_{m_p} &= pn + \epsilon_{M_p}, \quad \varepsilon_{m_p} = pn + \varepsilon_{M_p}, \quad \epsilon_{m_p} = \varepsilon_{m_p} + p + 1, \\ m_p(*|z_1 \dots z_i, z_{i+1} \dots z_p) &= (-1)^n m_p(*|z_1 \dots z_{i+1}^*, z_i^* \dots z_p). \end{aligned} \quad (5.4)$$

Here  $z^* = (x, -\xi)$  if  $z = (x, \xi)$ .

Introduce the space  $\mathcal{M}_1 \subset C_2(\mathbf{D}_n, \mathbf{D}'_n)$  consisting of all 2-forms which can be locally represented as

$$M_{2|2}^1(z|f, g) = \sum_{q=0}^Q m^{1(A)q}(z|[(\partial_A^z)^q f(z)]g - (-1)^{\epsilon(f)\epsilon(g)}[(\partial_A^z)^q g(z)]f), \quad (5.5)$$

with locally constant  $Q$  and the space  $\mathcal{M}_2 \subset C_2(\mathbf{D}_n, \mathbf{D}'_n)$  consisting of all 2-forms which can be locally represented as

$$M_{2|2}^2(z|f, g) = \sum_{q=0}^Q m^{2(A)q}(z|[(\partial_A)^q f]g - (-1)^{\epsilon(f)\epsilon(g)}(\partial_A)^q g]f) \quad (5.6)$$

with locally constant  $Q$ , where  $m^{1,2(A)q}(z|\cdot) \in C_1(\mathbf{D}_n, \mathbf{D}'_n)$ .

The space  $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$  is called in this paper the space of local bilinear forms. It consists of all the form, which can be present as

$$M_{2|\text{loc}}(z|f, g) = \sum_{p, q=0}^Q m^{(A)q|(B)p}(z) ((\partial_A)^q f(z) (\partial_B)^p g(z) - (-1)^{\epsilon(f)\epsilon(g)} (\partial_A)^q g(z) (\partial_B)^p f(z)).$$

Here  $m^{(A)q|(B)p} \in D' \otimes \mathbb{G}^n$ , and the summation limit  $Q$  is locally constant with respect to  $z$ .

## 6 $H_E^2$ for $n = 1$ antibracket

Here we give the proof of the point b) in Theorem 3.1.

**Proposition 6.1.** *Let  $n = 1$ .*

*Let the bilinear form*

$$M_2(z|f, g) = \int dv du m_2(z|u, v) f(u) g(v),$$

such that  $M_2(z|f, g) \in \mathbf{E}_1$  for all  $f, g \in \mathbf{D}_1$  be cocycle, i.e. it satisfy the cohomology equation

$$\begin{aligned} d_2^{\text{ad}} M_2(z|f, g, h) = & -(-1)^{\epsilon(f)\epsilon(h)} \{ (-1)^{\epsilon(f)\epsilon(h)} [M_2(z|f, g), h(z)] + \\ & + (-1)^{\epsilon(f)\epsilon(h)} M_2(z|[f, g], h) + \text{cycle}(f, g, h) \} = 0. \end{aligned} \quad (6.1)$$

Then

$$\begin{aligned} M_2(z|f, g) = & c_1 m_{2|1}(x|f, g) + c_2 m_{2|2}(x|f, g) + c_5 m_{2|5}(x|f, g) + \\ & + c_6 m_{2|6}(x|f, g) + d_1^{\text{ad}} M_{1|1}(z|f, g) + M_{2\text{loc}}(z|f, g). \end{aligned}$$

where  $c_i$  are constants,  $m_{2|i}$  are defined in Theorem 3.1 and  $M_{2\text{loc}}(z|f, g) \in \mathcal{M}_0$ .

The details of the proof can be found in Appendix 1.

The space of local cocycles is generated up to coboundaries by  $m_{2|3}$  (odd cocycle) and  $m_{2|4}$  (even cocycle) [9].

## 7 Deformation with one even and one odd parameter

Let

$$\begin{aligned} [f(z), g(z)]_* = & A(z|f, g; \hbar, \theta) = A_1(z|f, g; \hbar) + \theta A_0(z|f, g; \hbar), \\ J_{A,A}(z|f, g, h) = & (-1)^{(\epsilon(f))(\epsilon(h))} A(z|A(f, g; \hbar, \theta), h; \hbar, \theta) + \text{cycle}(f, g, h) = 0, \end{aligned} \quad (7.1)$$

where  $\varepsilon_{\hbar} = 0$ ,  $\varepsilon_{A_1} = 1$ ,  $\varepsilon_{\theta} = 1$ ,  $\varepsilon_{A_0} = 0$ .

It follows from Jacobi identity (7.1):

$$J_{A_1, A_1}(z|f, g, h) = 0, \quad (7.2)$$

$$J_{A_1, \theta A_0}(z|f, g, h) = 0, \quad (7.3)$$

such that we have from Theorem 4.1

$$A_1[f, g; \hbar] = [f(z), g(z)] + (-1)^{\varepsilon(f)} \left\{ \frac{\hbar c}{1 + \hbar c N_z / 2} \Delta f(z) \right\} \mathcal{E}_z g(z) + \{ \mathcal{E}_z f(z) \} \frac{\hbar c}{1 + \hbar c N_z / 2} \Delta g(z) \quad (7.4)$$

(up to similarity transformation of  $[f(z), g(z)]_*$ )

If  $A_0 \neq 0$  then we can redefine  $\theta \mapsto \hbar^{-k} \theta$  with some definite  $k$  in such a way that the decomposition of  $A_0(z|f, g; \hbar)$  starts with zero degree of  $\hbar$ :  $A_0(z|f, g; 0) \neq 0$ .

Then (7.3) gives  $J_{A_1, \theta A_0}(z|f, g, 0) = 0$ , i.e.  $A_0|_{\hbar=0}$  is a cocycle, and since it is odd,  $A_0(z|f, g; 0) = c_{0|0} m_{2|3} z|f, g$  up to equivalence transformation.



To prove Theorem 4 it remains to prove that if  $A_0 \neq 0$  then  $A_1(z|f, g; \hbar) = [f, g]$ .  
Let us assume that  $A_0 \neq 0$ . Then we may assume that

$$A_0(z|f, g; \hbar) = \sum_{k_0=0}^{\infty} A_{0|k_0}(z|f, g), \quad A_{0|0}(z|f, g) \neq 0.$$

Let

$$A_1(z|f, g) = [f(z), g(z)] + \hbar^{k_1+1} c_{1|k_1} m_{2|4}(z|f, g) + O(\hbar^{k_1+2}).$$

Define the notation

$$A_{0|[m,n]}(z|f, g) = \sum_{l=m}^n \hbar^l A_{0|l}(z|f, g).$$

Let

$$c_1 = O(\hbar^{k_1}),$$

where  $k_1$  is some integers.  $k_1 \geq 1$ .

### 7.0.1 0-th, ... , $(k_1)$ -th orders in $\hbar$

In these cases, we find

$$d_2^{\text{ad}} A_{0|[0,k_1]}(z|f, g, h) = 0,$$

such that we obtain (up to similarity transformation)

$$A_{0|[0,k_1-1]}(z|f, g, h) = c_{0|[0,k_1]} m_{2|3}(z|f, g), \quad c_{0|0} \neq 0.$$

Here  $c_{0|[m,n]} = \sum_{k=m}^n c_{0|k}$ ,  $c_{0|0} \neq 0$ .

Before we will start to consider remaining case let us formulate the following proposition

**Proposition 7.1.** *Let*

$$\begin{aligned} & (-1)^{(\epsilon(f))(\epsilon(h))} [[A(z|f, g), h(z)] + A(z|[f, g], h)] + \text{cycle}(f, g, h) + \\ & + c J_{m_{2|3}, m_{2|4}}(z|f, g, h) = 0. \end{aligned} \tag{7.5}$$

for some  $c \in \mathbb{K}$  and some  $A \in C_2(\mathbf{D}_n, \mathbf{D}_n)$

Then  $c = 0$ .

Proof.

1. Note that up to some similarity transformation  $A$  is local form,  $A \in \mathcal{M}_0$ .

Indeed, consider the domains

i)  $z \cap [\text{supp}(f) \cup \text{supp}(g) \cup \text{supp}(h)] = \text{supp}(f) \cap [\text{supp}(g) \cup \text{supp}(h)] = \emptyset$

and

ii)  $z \cap [\text{supp}(f) \cup \text{supp}(g) \cup \text{supp}(h)] = \emptyset$

In these domains,  $J_{m_{2|3}, m_{2|4}}(z|f, g, h) = 0$  and, as it is shown in [9],  $A(z|f, g)$  can be represented in the form

$$\begin{aligned}
A(z|f, g) &= A_{\text{loc}}(z|f, g) + d_1^{\text{ad}} M(z|f, g) \\
A_{\text{loc}}(z|f, g) &= \sum_{a,b=0}^N (-1)^{\varepsilon(f)|\varepsilon_B|1,b+1} m^{(A)_a|(B)_b}(z) [(\partial_A^z)^a f(z)] (\partial_B^z)^b g(z) \\
m^{(B)_b|(A)_a} &= (-1)^{|\varepsilon_A|1,a|\varepsilon_B|1,b} m^{(A)_a|(B)_b}, \quad \varepsilon(m^{(A)_a|(B)_b} (\partial_A^z)^a (\partial_B^z)^b) = 0,
\end{aligned}$$

where  $M(z|f)$  is some 1-form,  $\varepsilon_M = 1$ .

After similarity transformation of  $[f, g]$  with  $T(z|f) = f(z) - \hbar \theta M(z|f)$  we have  $A(z|f, g) = A_{\text{loc}}(z|f, g)$  and  $A_{\text{loc}}(z|f, g)$  satisfies eq. (7.5).

Choosing  $f(z) = e^{zp}$ ,  $g(z) = e^{zq}$ ,  $h(z) = e^{zr}$  in some neighbourhood of  $z$ , we reduce eq. (7.5) to the form

$$\begin{aligned}
\Phi(z|p, q, r) \langle p, q \rangle - [F(z|p, q), zr] + \text{cycle}(p, q, r) = \\
= c \cdot (m_{2|3}(z|m_{2|4}; p, q, r) + m_{2|4}(z|m_{2|3}; p, q, r) + \text{cycle}(p, q, r))
\end{aligned} \tag{7.6}$$

where

$$\begin{aligned}
F(z|p, q) &= \sum_{a,b=0}^N (-1)^{\varepsilon(f)|\varepsilon_B|1,b+1} m^{(A)_a|(B)_b}(z) (p_A)^a (q_B)^b = F(z|q, p) = \\
&= m^{0|0}(z) + m^A(z)(p_A + q_A) + O((\text{momenta})^2), \\
m^A(z) &= m^{0|A}(z) = m^{A|0}(z), \\
\langle p, q \rangle &= [e^{zp}, e^{zq}] e^{-z(p+q)} \\
\Phi(z|p, q, r) &= F(z|p+q, r) - F(z|p, r) - F(z|q, r) = \\
&= -m^{0|0}(z) + O(\text{momenta}) \\
m_{2|3}(z|m_{2|4}; p, q, r) &= m_{2|3}(z|m_{2|4}(|e^{zp}, e^{zq}), e^{zr}) e^{-z(p+q+r)} = \\
&= -\frac{1}{2} \{ [\langle p, p \rangle (1 - zp/2) + \langle q, q \rangle (1 - zp/2)] (1 - \xi\alpha - \xi\beta) + \\
&\quad + \langle p, p \rangle \xi\beta/2 + \langle q, q \rangle \xi\alpha/2 \} (1 - \xi\gamma) \\
m_{2|4}(z|m_{2|3}; p, q, r) &= m_{2|4}(z|m_{2|3}(|e^{zp}, e^{zq}), e^{zr}) e^{-z(p+q+r)} = \\
&= \{ \xi\alpha(\xi\beta - 1)(u\alpha + v\alpha) + \xi\beta(\xi\alpha - 1)(u\beta + v\beta) \} (1 - zr/2) + \\
&\quad + \frac{1}{2} \left\{ 1 - \frac{1}{2}\xi\alpha - \frac{1}{2}\xi\beta - \frac{1}{2}(1 - \xi\alpha)(1 - \xi\beta)(zp + zq) \right\} \langle r, r \rangle.
\end{aligned}$$

Here

$$zp = z^A p_A, \quad p_A = (u_i, \alpha_i), \quad q_A = (v_i, \beta_i), \quad r_A = (t_i, \gamma_i), \quad \xi\alpha = \xi^i \alpha_i, \quad u\alpha = u_i \alpha_i$$

and so on.

For  $r = 0$ , we find

$$\begin{aligned}
[m_{2|3}(z|m_{2|4}; p, q, r) + m_{2|4}(z|m_{2|3}; p, q, r) + \text{cycle}(p, q, r)]|_{r=0} &= P_4(p, q) \\
P_4(p, q) &= -u\alpha - v\beta + (u\alpha)(\xi\beta) + (v\beta)(\xi\alpha) - (u\beta)(\xi\beta) - (v\alpha)(\xi\alpha) + \\
&\quad + (u\beta)(\xi\alpha)(\xi\beta) + (v\alpha)(\xi\alpha)(\xi\beta)
\end{aligned}$$

At  $r = 0$ , eq. (7.6) takes the form

$$\begin{aligned} \Phi(z|p, q)\langle p, q \rangle - [F(z|p), zq] - [F(z|q), zp] &= c P_4(p, q), \\ \Phi(z|p, q) &= F(z|p+q) - F(z|p) - F(z|q), \quad F(z|p) = F(z|p, 0). \end{aligned} \quad (7.7)$$

Consider in eq. (7.7) the terms of the second order in momenta. We obtain the reduced equation

$$m^{0|0}(z)\langle p, q \rangle + [m^A(z)q_A, zp] + [m^A(z)p_A, zq] = c(u\alpha + v\beta)$$

which implies

$$c = 0.$$

Q.E.D.

### 7.0.2 $(k_1 + 1)$ -th order in $\hbar$

In this case, we find

$$\begin{aligned} (-1)^{(\epsilon(f))(\epsilon(h))} [A_{0|k_1+1}(z|f, g), h(z)] + A_{0|k_1+1}(z|[f, g], h) + \text{cycle}(f, g, h) + \\ + c_{0|0}c_{1|k_1}J_{m_{2|3}, m_{2|4}}(z|f, g, h) = 0. \end{aligned} \quad (7.8)$$

It follows from eq. (7.8) and Proposition 7.1 that

$$c_{0|0}c_{1|k_1} = 0.$$

and so  $c_{1|k_1} = 0$ .

Using the induction method, we obtain that if  $A_0 \neq 0$  then the general solution of eq. (7.1) (up to similarity transformation) is

$$[f(z), g(z)]_* = [f(z), g(z)] + \theta A_0(z|f, g) = [f(z), g(z)] + \theta \sum_i c_{0|i} \hbar^i m_{2|3}(z|f, g),$$

or after redefining  $\theta$

$$[f(z), g(z)]_* = [f(z), g(z)] + \theta m_{2|3}(z|f, g).$$

## Appendix 1. $H_E^2$ for $n = 1$ antibracket

### 1.1. General solution

Let  $\epsilon = \varepsilon + 1$ . The conomology equation for antibracket can be represented in the form

$$\begin{aligned} d_2^{\text{ad}} M_2(f, g, h) = -(-1)^{\epsilon(f)\epsilon(h)} J_{M_2, m_0}(z|f, g, h) = \\ -(-1)^{\epsilon(f)\epsilon(h)} ((-1)^{\epsilon(f)\epsilon(h)} [[M_2(z|f, g), h(z)] + M_2(z|[f, g], h) + \text{cycle}(f, g, h)) = 0. \end{aligned} \quad (\text{A1.1})$$

Introduce notation:

$$f(z) = f_0(x) + \xi f_1(x) = \check{f}_0(z) + \check{f}_1(z), \quad \check{f}_0(z) = f_0(x), \quad \check{f}_1(z) = \xi f_1(x).$$

Represent the forms  $M_1(z|f)$  and  $M_2(z|f, g)$  in the form

$$M_1(z|f) = T_{(1)}(x|f_0) + T_{(2)}(x|f_1) + \xi[T_{(3)}(x|f_0) + T_{(4)}(x|f_1)],$$

$$\begin{aligned} M_2(z|f, g) &= M_{(1)}(x|f_0, g_0) + M_{(2)}(x|f_0, g_1) - M_{(2)}(x|g_0, f_1) + M_{(3)}(x|f_1, g_1) + \\ &+ \xi[M_{(4)}(x|f_0, g_0) + M_{(5)}(x|f_0, g_1) - M_{(5)}(x|g_0, f_1) + M_{(6)}(x|f_1, g_1)], \\ M_{(1,4)}(x|\varphi, \phi) &= M_{(1,4)}(x|\phi, \varphi), \quad M_{(3,6)}(x|\varphi, \phi) = -M_{(3,6)}(x|\phi, \varphi). \end{aligned}$$

We have for  $M_{2d}(z|f, g) = d_1^{\text{ad}} M_1(f, g)$ :

$$M_{d(1)}(x|\varphi, \phi) = -T_{(3)}(x|\varphi)\partial_x\phi(x) - T_{(3)}(x|\phi)\partial_x\varphi(x), \quad M_{d(4)}(x|\varphi, \phi) = 0, \quad (\text{A1.2})$$

$$M_{d(2)}(x|\varphi, \phi) = \partial_x T_{(1)}(x|\varphi)\phi(x) + T_{(4)}(x|\phi)\partial_x\varphi(x) - T_{(1)}(x|[\varphi, \phi]_0), \quad (\text{A1.3})$$

$$M_{d(3)}(x|\varphi, \phi) = \partial_x T_{(2)}(x|\varphi)\phi(x) - \partial_x T_{(2)}(x|\phi)\varphi(x) - T_{(2)}(x|[\varphi, \phi]_1), \quad (\text{A1.4})$$

$$M_{d(5)}(x|\varphi, \phi) = \partial_x T_{(3)}(x|\varphi)\phi(x) - T_{(3)}(x|\varphi)\partial_x\phi(x) - T_{(3)}(x|[\varphi, \phi]_0), \quad (\text{A1.5})$$

$$\begin{aligned} M_{d(6)}(x|\varphi, \phi) &= \partial_x T_{(4)}(x|\varphi)\phi(x) - \partial_x T_{(4)}(x|\phi)\varphi(x) + T_{(4)}(x|\phi)\partial_x\varphi(x) - \\ &- T_{(4)}(x|\varphi)\partial_x\phi(x) - T_{(4)}(x|[\varphi, \phi]_1), \end{aligned} \quad (\text{A1.6})$$

$$[\varphi(x), \phi(x)]_0 = \{\partial_x\varphi(x)\}\phi(x), \quad [\varphi(x), \phi(x)]_1 = \{\partial_x\varphi(x)\}\phi(x) - \varphi(x)\partial_x\phi(x).$$

It follows from  $J_{M_2, m_0}(z|\check{f}_0, \check{g}_0, \check{h}_0) = 0$  and  $J_{M_2, m_0}(z|\check{f}_0, \check{g}_0, \check{h}_1) = 0$  that

$$M_{(4)}(x|\varphi, \phi)\partial_x\omega(x) + \text{cycle}(\varphi, \phi, \omega) = 0, \quad (\text{A1.7})$$

$$M_{(4)}(x|\varphi, \phi)\partial_x\omega(x) - \{\partial_x M_{(4)}(x|\varphi, \phi)\}\omega(x) + M_{(4)}(x|[\varphi, \omega]_0, \phi) + M_{(4)}(x|\varphi, [\phi, \omega]_0) = 0,$$

$$\begin{aligned} &M_{(1)}(x|[\varphi, \omega]_0, \phi) + M_{(1)}(x|\varphi, [\phi, \omega]_0) - \{\partial_x M_{(1)}(x|\varphi, \phi)\}\omega(x) - \\ &- M_{(5)}(x|\varphi, \omega)\partial_x\phi(x) - M_{(5)}(x|\phi, \omega)\partial_x\varphi(x) = 0. \end{aligned} \quad (\text{A1.8})$$

It follows from  $d_2^{\text{ad}} M_2(z|\check{f}_0, \check{g}_1, \check{h}_1) = 0$  that

$$\begin{aligned} &M_{(5)}(x|\varphi, \phi)\partial_x\omega(x) - \{\partial_x M_{(5)}(x|\varphi, \phi)\}\omega(x) + \{\partial_x M_{(5)}(x|\varphi, \omega)\}\phi(x) - \\ &- M_{(5)}(x|\varphi, \omega)\partial_x\phi(x) - M_{(5)}(x|[\varphi, \phi]_0, \omega) + M_{(5)}(x|[\varphi, \omega]_0, \phi) + M_{(5)}(x|\varphi, [\phi, \omega]_1) = 0. \end{aligned} \quad (\text{A1.9})$$

$$\begin{aligned} &\{\partial_x M_{(2)}(x|\varphi, \omega)\}\phi(x) - \{\partial_x M_{(2)}(x|\varphi, \phi)\}\omega(x) - M_{(2)}(x|[\varphi, \phi]_0, \omega) + \\ &+ M_{(2)}(x|[\varphi, \omega]_0, \phi) + M_{(2)}(x|\varphi, [\phi, \omega]_1) + M_{(6)}(x|\phi, \omega)\partial_x\varphi(x) = 0. \end{aligned} \quad (\text{A1.10})$$

It follows from  $d_2^{\text{ad}} M_2(z|\check{f}_1, \check{g}_1, \check{h}_1) = 0$  that

$$\begin{aligned} &M_{(6)}(x|\varphi, \phi)\partial_x\omega(x) - \{\partial_x M_{(6)}(x|\varphi, \phi)\}\omega(x) - M_{(6)}(x|[\varphi, \phi]_1, \omega) + \\ &+ \text{cycle}(\varphi, \phi, \omega) = 0, \end{aligned} \quad (\text{A1.11})$$

$$- [\{\partial_x M_{(3)}(x|\varphi, \phi)\}\omega(x) + M_{(3)}(x|[\varphi, \phi]_1, \omega) + \text{cycle}(\varphi, \phi, \omega)] = 0. \quad (\text{A1.12})$$

I. Consider Eq. (A1.7). As it was shown in [9], we find

$$M_{(4)}(x|\varphi, \phi) = 0.$$

II. Consider Eq. (A1.9). As it was shown in [9], we find

$$M_{(5)}(x|\varphi, \phi) = M_{d|(5)}(x|\varphi, \phi) + \text{loc},$$

where the expression for  $M_{d|(5)}(x|\varphi, \phi)$  is given by Eq. (A1.5).

III. Consider Eq. (A1.8). As it was shown in [9], we find

$$M_{(1)}(x|\varphi, \phi) = M_{d|(1)}(x|\varphi, \phi) + \text{loc},$$

where (the expression for  $M_{d|(1)}(x|\varphi, \phi)$  is given by Eq. (A1.2).

IV. Consider Eq. (A1.11). As it was shown in [9], we find

$$M_{(6)}(x|\varphi, \phi) = M_{d|(6)}(x|\varphi, \phi) + \text{loc},$$

where the expression for  $M_{d|(6)}(x|\varphi, \phi)$  is given by Eq. (A1.6).

V. Consider Eq. (A1.10). As it was shown in [9], we find

$$\begin{aligned} M_{(2)}(x|\varphi, \phi) &= M_{(2)8}(x|\varphi, \phi) + M_{d|(2)}(x|\varphi, \phi) + \text{loc}, \\ M_{(2)8}(x|\varphi, \phi) &= \sum_{q=0, q \neq 1}^Q M_7^q(x|\{\partial^q \varphi\} \phi), \quad \partial_x \hat{M}_7^q(x|\varphi) = 0, \end{aligned}$$

where the expression for  $M_{d|(2)}(x|\varphi, \phi)$  is given by Eq. (A1.3).

For  $M_{(2)8}(x|\varphi, \phi)$  we obtain an equation

$$\begin{aligned} &\{\partial_x M_{(2)8}(x|\varphi, \omega)\} \phi(x) - \{\partial_x M_{(2)8}(x|\varphi, \phi)\} \omega(x) - M_{(2)8}(x|[\varphi, \phi]_0, \omega) + \\ &+ M_{(2)8}(x|[\varphi, \omega]_0, \phi) + M_{(2)8}(x|\varphi, [\phi, \omega]_1) = \text{loc}. \end{aligned}$$

Let

$$x \cap [\text{supp}(\varphi) \cup \text{supp}(\phi) \cup \text{supp}(\omega)] = \emptyset.$$

We obtain

$$\hat{M}_{(2)8}(x|[\varphi, \phi]_0, \omega) - \hat{M}_{(2)8}(x|[\varphi, \omega]_0, \phi) - \hat{M}_{(2)8}(x|\varphi, [\phi, \omega]_1) = 0$$

or

$$\sum_{q=0, q \neq 1}^Q \hat{M}_7^q(x|\{\partial^q(\partial\varphi\phi)\}\omega - \{\partial^q(\partial\varphi\omega)\}\phi - \{\partial^q\varphi\}[\partial\phi\omega - \phi\partial\omega]) = 0. \quad (\text{A1.13})$$

Let  $\varphi(x) \rightarrow e^{px}\varphi(x)$  and  $\phi(x) \rightarrow e^{kx}$ ,  $\omega(x) \rightarrow e^{-(p+k)x}$  for  $x \in \text{supp}\varphi$ .

Consider the terms of highest order in  $p, k$  in Eq. (A1.13),

$$[p(p+k)^Q - p(-k)^Q - (p+2k)p^Q] \hat{M}_7^Q(x|\varphi) = 0 \implies$$

$$\begin{aligned} \hat{M}_7^q(x|\varphi) = 0, \quad q \neq 0, 2 &\implies M_{(2)8}(x|\varphi, \phi) = M_7^2(x|\{\partial^2\varphi\}\phi) + M_7^0(x|\varphi\phi) + \text{loc}, \\ \partial \hat{M}_7^2(x|\varphi) = \partial \hat{M}_7^0(x|\varphi) &= 0. \end{aligned}$$

Consider the terms of the second order in  $p, k$  in eq. (A1.13) (the terms of third order are identically cancelled),

$$(p^2 + 2pk)\hat{M}_7^2(x|\partial\varphi) = 0 \implies \partial_x \hat{m}_7^2(x|y) = \hat{m}_7^2(x|y)\overleftarrow{\partial}_y = 0 \implies m_7^2(x|y) = c_5 + 2c_6\theta(x-y) + \text{loc},$$

where

$$\begin{aligned} M_7^2(x|\varphi) &= \int dy m_7^2(x|y)\varphi(y) = c_5 \int dy \varphi(y) + c_6 \int dy \theta(x-y)\varphi(y) + \text{loc} \implies \\ M_{(2)8}(x|\varphi, \phi) &= c_5 \int dy \{\partial^2 \varphi(y)\}\phi(y) + \\ &+ 2c_6 \int dy \theta(x-y)\{\partial^2 \varphi(y)\}\phi(y) + M_7^0(x|\varphi\phi) + \text{loc}. \end{aligned}$$

It follows from eq. (A1.13)

$$\hat{M}_7^0(x|\varphi\partial\phi\omega - \varphi\phi\partial\omega) = 0 \implies M_7^0(x|\varphi) = \text{loc}.$$

Finally, we have

$$\begin{aligned} M_{(2)}(x|\varphi, \phi) &= c_5 \tilde{\mu}_{2|5}(x|\varphi, \phi) + 2c_6 \tilde{\mu}_{2|6}(x|\varphi, \phi) + M_{d|(2)}(x|\varphi, \phi) + \text{loc}, \\ \tilde{\mu}_{2|5}(x|\varphi, \phi) &= \int dy \{\partial^2 \varphi(y)\}\phi(y), \quad \tilde{\mu}_{2|6}(x|\varphi, \phi) = \int dy \theta(x-y)\{\partial^2 \varphi(y)\}\phi(y), \end{aligned}$$

or, after equivalent transformations and notation changing

$$\begin{aligned} M_{(2)}(x|\varphi, \phi) &= c_5 \mu_{2|5}(x|\varphi, \phi) + c_6 \mu_{2|6}(x|\varphi, \phi) + M_{d|(2)}(x|\varphi, \phi) + \text{loc}, \\ \mu_{2|5}(x|\varphi, \phi) &= \int dy \{\partial \varphi(y)\}\partial \phi(y), \quad \mu_{2|6}(x|\varphi, \phi) = \int dy \theta(x-y)\{\partial \varphi(y)\}\partial_y \phi(y), \end{aligned}$$

VI. Consider Eq. (A1.12). As it was shown in [9], we find

$$\begin{aligned} M_{(3)}(x|\varphi, \phi) &= c_1 \mu_{2|1}(x|\varphi, \phi) + c_2 \mu_{2|2}(x|\varphi, \phi) + M_{d|(3)}(x|\varphi, \phi) + \text{loc}, \\ \mu_{2|1}(x|\varphi, \phi) &= \int dy [\partial_y^3 \varphi(y)]\phi(y), \\ \mu_{2|2}(x|\varphi, \phi) &= \int dy \theta(x-y)[\{\partial_y^3 \varphi(y)\}\phi(y) - \varphi(y)\partial_y^3 \phi(y)]. \end{aligned}$$

Introduce two forms  $m_{2|a}(z|f, g)$ ,  $a = 1, 2$ ,  $\epsilon_{m_{2|a}} = 1$ ,

$$\begin{aligned} m_{2|1}(z|f, g) &= \int du (-1)^{\epsilon(f)} [\partial_y^3 f(u)] \partial_\eta g(u), \\ m_{2|2}(z|f, g) &= \int du \theta(x-y) (-1)^{\epsilon(f)} \{ [\partial_y^3 f(u)] \partial_\eta g(u) + (-1)^{\epsilon(g)} [\partial_\eta f(u)] \partial_y^3 g(u) \} - \\ &- x \{ [\partial_x^2 \partial_\xi f(z)] \partial_x \partial_\xi g(z) - [\partial_x \partial_\xi f(z)] \partial_x^2 \partial_\xi g(z) \}. \end{aligned}$$

These forms have the properties:

$$\begin{aligned}
m_{2|a}(z|\check{f}_0, \check{g}_0) &= m_{2|a}(z|\check{f}_1, \check{g}_0) = m_{2|a}(z|\check{f}_0, \check{g}_1) = 0, \\
m_{2|1}(z|\check{f}_1, \check{g}_1) &= \int dy [\partial^3 f_1(y)] g_1(y) = \mu_{2|1}(x|f_1, g_1), \\
m_{2|2}(z|\check{f}_1, \check{g}_1) &= \int dy \theta(x-y) [\{\partial_y^3 f_1(y)\} g_1(y) - f_1(y) \partial_y^3 g_1(y)] - \\
&\quad - x [\{\partial_x^2 f_1(x)\} \partial_x g_1(x) - \{\partial_x f_1(x)\} \partial_x^2 g_1(x)] = \mu_{2|2}(x|f_1, g_1) + \text{loc.}
\end{aligned}$$

$$\begin{aligned}
m_{2|a}(z|g, f) &= -(-1)^{\epsilon(f)\epsilon(g)} m_{2|a}(z|f, g), \\
d_2^{\text{ad}} m_{2|a}(z|f, g, h) &= 0.
\end{aligned}$$

Introduce two forms  $m_{2|a}(z|f, g)$ ,  $a = 5, 6$ ,  $\epsilon_{m_{2|a}} = 0$ ,

$$\begin{aligned}
m_{2|5}(z|f, g) &= \int du (-1)^{\epsilon(f)} \partial_y f(u) \partial_y g(u), \\
m_{2|6}(z|f, g) &= \int du \theta(x-y) (-1)^{\epsilon(f)} \partial_y f(u) \partial_y g(u).
\end{aligned}$$

These forms have the properties:

$$\begin{aligned}
m_{2|a}(z|\check{f}_0, \check{g}_0) &= m_{2|a}(z|\check{f}_1, \check{g}_1) = 0, \\
m_{2|5}(z|\check{f}_0, \check{g}_1) &= \int dy [\partial_y f_0(y)] \partial_y g_1(y) = \mu_{2|5}(x|f_0, g_1), \\
m_{2|6}(z|\check{f}_0, \check{g}_1) &= \int dy \theta(x-y) [\partial_y f_0(y)] \partial_y g_1(y) = \mu_{2|6}(x|f_0, g_1).
\end{aligned}$$

$$\begin{aligned}
m_{2|a}(z|g, f) &= -(-1)^{\epsilon(f)\epsilon(g)} m_{2|a}(z|f, g), \\
d_2^{\text{ad}} m_{2|a}(z|f, g, h) &= 0.
\end{aligned}$$

So, we obtained

$$\begin{aligned}
M_2(z|f, g) &= c_1 m_{2|1}(x|f, g) + c_2 m_{2|2}(x|f, g) + c_5 m_{2|5}(x|f, g) + \\
&\quad + c_6 m_{2|6}(x|f, g) + d_1^{\text{ad}} M_{1|1}(z|f, g) + M_{2\text{loc}}(z|f, g).
\end{aligned}$$

The local form  $M_{2\text{loc}}(z|f, g)$  satisfies the equation  $d_2^{\text{ad}} M_{2\text{loc}}(z|f, g, h) = 0$ , the solution of which, as it was shown in [9], is

$$\begin{aligned}
M_{2\text{loc}}(z|f, g) &= c_3 m_{2|3}(x|f, g) + c_4 m_{2|4}(x|f, g) + d_1^{\text{ad}} M_{1|2}(z|f, g), \\
m_{2|3}(x|f, g) &= (-1)^{\epsilon(f)} \{(1 - N_\xi) f(z)\} (1 - N_\xi) g(z), \quad \epsilon_{m_{2|3}} = 0, \\
m_{2|4}(x|f, g) &= (-1)^{\epsilon(f)} [\Delta f(z)] \hat{l}_z g(z) + [\hat{l}_z f(z)] \Delta g(z), \quad \epsilon_{m_{2|4}} = 1.
\end{aligned}$$

Finally, we find: general solution of eq. (A1.1) is

$$M_2(z|f, g) = \sum_1^6 c_i m_{2|i}(x|f, g) + d_1^{\text{ad}} M_1(z|f, g).$$

## 1.2. Adjoint Cohomology

Let  $D$  denotes  $\mathbf{D}_1$ . We will say that the form  $M(f, g, \dots)$  is compact and we will write  $M = \text{comp}$  if  $M(f, g, \dots) \in D$  for any  $f, g, \dots \in D$ .

Here we prove an useful

**Statement** The form  $M_2(z|f, g)$  is compact iff  $c_1 = c_2 = c_5 = c_6 = 0$ . and  $d_1^{\text{ad}} M_1(z|f, g) = \text{comp}$ .

Proof.

We must solve the equations

$$c_1 m_{2|1}(x|f, g) + c_2 m_{2|2}(x|f, g) + M_1(z|\{f, g\}) = \text{comp}, \quad \varepsilon_{M_1} = 1, \quad (\text{A1.14})$$

$$c_5 m_{2|5}(x|f, g) + c_6 m_{2|6}(x|f, g) + M_1(z|\{f, g\}) = \text{comp}, \quad \varepsilon_{M_1} = 0. \quad (\text{A1.15})$$

First, consider eq. (A1.14).

It must be  $\varepsilon_{M_1} = 1$ , such that we have

$$M_1(z|f) = \int du m_1(z|u) f(u), \quad m_1(z|u) = \mu(x|y) + \xi \eta \nu(x|y) \implies$$

$$M_1(z|f) = \int dy^0 \mu(x|y) f_1(y) - \xi \int dy \nu(x|y) f_0(y),$$

and

$$\begin{aligned} c_1 m_{2|1}(z|f, g) + c_2 m_{2|2}(z|f, g) + \int dy \mu(x|y) [f'_1(y) g_1(y) - f_1(y) g'_1(y)] = \\ = \text{comp}, \end{aligned} \quad (\text{A1.16})$$

$$\int dy \nu(x|y) [f'_0(y) g_1(y) - f_1(y) g'_0(y)] = \text{comp}. \quad (\text{A1.17})$$

Consider eq. (A1.17). Choosing  $g_0(y) = y$  for  $y \in \text{supp} f_1$  and  $f_0(y) = 1$  for  $y \in \text{supp} g_1$ , we obtain

$$\int dy \nu(x|y) f(y) = \text{comp}, \quad \forall f \in D.$$

Turn to eq. (A1.16).

i) Choosing  $f_1(y) = 1$  for  $y \in \text{supp} g_1$ , we obtain

$$\begin{aligned} \int dy \mu(x|y) g'(y) = \text{comp}, \quad \forall g \in D \implies \\ \implies c_1 m_{2|1}(z|f, g) + c_2 m_{2|2}(z|f, g) - 2 \int dy \mu(x|y) f_1(y) g'_1(y) = \text{comp}. \end{aligned}$$



Further choosing  $g_0(y) = y$  for  $y \in \text{supp} f_1$ , we find finally

$$\int du^0 \mu(x|u^0) f(u^0) = \text{comp}, \quad \forall f \in D,$$

and as a consequence

$$c_1 m_{2|1}(z|f, g) + c_2 m_{2|2}(z|f, g) = \text{comp}.$$

Let  $x \rightarrow -\infty$ . We obtain  $c_1 m_{2|1}(z|f, g) = \text{comp} \implies c_1 = 0 \implies c_2 = 0$ .

Now, consider eq. (A1.15).

Since  $\varepsilon_{M_1} = 0$ , we have

$$M_1(z|f) = \int dy^0 \mu(x|y) f_0(y) - \xi \int dy \nu(x|y) f_1(y),$$

and

$$c_5 \mu_{2|5}(x|f_0, g_1) + c_6 \mu_{2|6}(x|f_0, g_1) + \int dy \mu(x|y) f'_0(y) g_1(y) = \text{comp}, \quad (\text{A1.18})$$

$$\int dy \nu(x|y) [f'_1(y) g_1(y) - f_1(y) g'_1(y)] = \text{comp}. \quad (\text{A1.19})$$

Setting  $g_1(y) = 1$  for  $y \in \text{supp} f_1$  in eq. (A1.19), we find  $\int dy \nu(x|y) f'_1(y) = \text{comp}$ ,  $\forall f_1(y) \in D \implies \int dy \nu(x|y) f_1(y) g'_1(y) = \text{comp}$ . Choosing  $g_1(y) = y$  for  $y \in \text{supp} f_1$ , we obtain

$$\int dy \nu(x|y) f(y) = \text{comp}, \quad \forall f(y) \in D.$$

Now, setting  $f_0(y) = y$  for  $y \in \text{supp} g_1$  in eq. (A1.18), we find

$$\begin{aligned} \int dy \mu(x|y) g(y) &= \text{comp}, \quad \forall g(y) \in D \implies \\ \implies c_5 \mu_{2|5}(x|f_0, g_1) + c_6 \mu_{2|6}(x|f_0, g_1) &= \text{comp} \implies c_5 = c_6 = 0. \end{aligned}$$

As a consequence, all forms  $m_{2|i}(z|f, g)$ ,  $i = 1, 2, \dots, 6$ , are independent nontrivial cohomology.

## References

- [1] *Batalin I.A., Vilkovisky G.A.*, Phys. Lett., **120B**, 166 (1983).
- [2] *I.A. Batalin and G.A. Vilkovisky*, J. Math. Phys., **26**, 172 (1985).
- [3] *Gomis J., Paris J., Samuel S.*, Antibrackets, antifields and gauge theory quantization, Phys. Rep., **259** 1–145 (1995).
- [4] *D.M. Gitman and I.V. Tyutin*, Quantization of Fields with Constraints, (Springer–Verlag, 1990).
- [5] *Henneaux M. and Teitelboim C.*, Quantization of Gauge Systems, Princeton University Press, Princeton, 1992.

- [6] *D. A. Leites and I. M. Shchepochkina*, How to quantize the antibracket, Theor. Math. Phys., **126**, 281–306 (2001).
- [7] *M. Scheunert and R. B. Zhang*, J.Math.Phys., **39**, 5024–5061 (1998); q-alg/9701037.
- [8] *S. E. Konstein, A. G. Smirnov and I. V. Tyutin*, Cohomologies of the Poisson superalgebra, Teor. Mat. Fiz., **143**, 625 (2005); hep-th/0312109.
- [9] *S. E. Konstein, and I. V. Tyutin*, Deformations and central extensions of the antibracket Superalgebra, Journal of Mathematical Physics, 49, 072103 (2008).
- [10] *S. E. Konstein, and I. V. Tyutin*, The deformations of nondegenerate constant Poisson bracket with even and odd deformation parameters, arXiv: 1001.1776 [hep-th]